PG – 586

III Semester M.Sc. Degree Examination, December 2015
(Y2K11 (RNS) Scheme)
MATHEMATICS
M 303 : Differential Geometry

Time : 3 Hours   Max. Marks : 80

Instructions : *1) Answer **any five** questions, choosing atleast **two** from **each** Part.*
*2) **All** questions carry **equal** marks.*

PART – A

1. a) Define directional derivatives of a real valued differentiable function on E^3.

   If $V_p = (v_1, v_2, v_3)_p$ is a tangent vector to E^3 at P and f is any real valued differentiable function on E^3, then prove that the $V_p[f] = \sum_{i=1}^{3} v_i \frac{\partial f}{\partial x_i}(P)$. Further compute $V_p[f]$ for $f = y^2z$ for give $V = (2, -1, 3)$, $P = (2, 0, -1)$. 9

   b) Show that every regular curve on E^3 has a unit speed reparametrization. Find unit speed reparametrization of the curve $\alpha(t) = (\cosh t, \sinh t, t)$. 7

2. a) Compute the derivative df of $f = xy^2 - yz^2$ and evaluate df on $V_p[f]$, where $V = (1, 2, -3)$, $P = (0, -2, 1)$. 4

   b) Let $\phi = xdx - ydy$ and $\theta = zdy$. Then find $\phi \wedge \theta$, verify the formula

   $d(\phi \wedge \theta) = d\phi \wedge \theta - \phi \wedge d\theta$. 5

   c) Let $F : E^3 \to E^3$; $F = (f_1, f_2, f_3)$ be a mapping. Prove that for any tangent vector V_p to E^3 at P.

   $F_{*p}(V_p) = (V_p[f_1], V_p[f_2], V_p[f_3])$ and deduce that F_{*p} is linear. Further compute $F_{*p}V_p$ for $F(x, y, z) = (x, xy, xyz)$ and $V = (1, -1, 3)$, $P = (2, 1, 0)$. 7

P.T.O.

3. a) Derive Frenet-formula for a unit speed curve. 5

b) Let β be a unit speed curve with curvature $K > 0$. Show that is the torsion τ of β is zero, then β is part of a circle of radius $1/K$. 5

c) With usual notations prove

$\nabla_{V_p} W = \sum V_p[w_i] U_i(P)$, $W = (w_1, w_2, w_3)$ $\nabla_{V_p} W$ of the vector field $W = xU_1 + x^2U_2 - x^2 U_3$ with respect to the tangent vector V_p : $V = (1, -1, 2)$, $P = (1, 3, -1)$. 6

4. a) Compute the connection forms of the frame field

$E_1 = \frac{1}{\sqrt{2}} (\sin f\, U_1 + U_2 - \cos f\, U_3)$

$E_2 = \frac{1}{\sqrt{2}} (\sin f\, U_1 - U_2 - \cos f\, U_3)$

$E_3 = \cos f\, U_1 + \sin f\, U_3$, for a function f. 5

b) Verify Carton's structural equations for cylindrical frame field. 5

c) Let $F : E^3 \to E^3$ be an isometry. Prove that there is a unique transformation T and a unique orthogonal transformation C, such that F = TC. 6

PART – B

5. a) Define a surface in E^3. Using the definition show that a unit sphere in E^3 is a surface in E^3. 5

b) If f is any real valued differentiable function on an open set D in E^2. Then show that the function $X : D \to E^3$:

$X(u, v) = (u, v, f(u, v))$, $\forall\, u, v \in D \subset E^2$ is a proper patch in E^3. 5

c) Let g be any real valued differentiable function on E^3 and C be a real number. Show that the set $M = \{(x, y, z) \in E^3 / g(x, y, z) = C\}$ is a surface in E^3 if $dg \neq 0$ at any point of M. 6

6. a) Let $M = \{(x, y, z) \in E^3 / g(x, y, z) = C\}$ be a surface in E^3. Prove that the gradient ∇g is a non-varnishing normal vector field on M. 8

b) Define pull back of a differential form. Show that is $X : D \subset E^2 \to M$ is a patch and ϕ and γ are respectively 1-form and 2-form on M, then

i) $X^*(\phi) = \phi(X_u)du + \phi(X_v)dv$ ii) $X^*(\gamma) = \gamma(X_u, X_v)\, du\, dv$. 8

7. a) If P is a umbilic point of a surface M in E^3, then prove that shape operator S of M is just a scalar multiple by $K = K_1 = K_2$. Further prove that if P is non umbilic then there are exactly two principal directions which are orthogonal. 8

b) Prove that the Gaussian and mean curvature of surface are given by

$$K = \frac{\begin{vmatrix} Sv.v & Sv.w \\ Sw.v & Sw.w \end{vmatrix}}{\begin{vmatrix} v.v & v.w \\ w.v & w.w \end{vmatrix}}$$

$$H = \frac{\begin{vmatrix} Sv.v & Sv.w \\ w.v & w.w \end{vmatrix} + \begin{vmatrix} v.v & v.w \\ Sw.v & Sw.w \end{vmatrix}}{2\begin{vmatrix} v.v & v.w \\ w.v & w.w \end{vmatrix}}$$ 8

8. a) Compute the Gaussian and mean curvature of

i) $X(u, v) = (u\cos v, u\sin v, bv)$, $b \neq 0$

ii) Saddle surface $z = xy$. 9

b) Compute the geodesics of (i) sphere (ii) cylinder 7